

# Some New Nyström Integrators

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National Aeronautics and  
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Houston, Texas





SHUTTLE PROGRAM

SOME NEW NYSTRÖM INTEGRATORS

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## SOME NEW NYSTRÖM INTEGRATORS

By William M. Lear  
TRW

### 1. INTRODUCTION

Nyström integrators are self-starting integrators used to integrate the second order, vector, differential equations  $\ddot{\underline{x}} = \underline{f}(t, \underline{x})$  and  $\ddot{\underline{x}} = \underline{f}(t, \underline{x}, \dot{\underline{x}})$ . Monuki, Reference 1, in a very extensive study has found the Nyström type integrators to be greatly superior to most Runge-Kutta type integrators in terms of speed and accuracy when integrating ballistic missile trajectories.

Nyström integrator parameters are defined by a set of nonlinear constraint equations. Frequently there are more parameters than there are equations. When this is the case, the investigator is free to choose some additional constraint equations. Additional equations are included for higher order (improved accuracy) integration when  $\ddot{\underline{x}} = \underline{f}(t)$ .

Two of the lower order integrators given here are not new, and are due to Nystrom, Reference 2. They are given so that the reader will have a single source for all orders of Nyström type integrators.

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## 2. TWO FUNCTION INTEGRATION OF $\ddot{x} = \underline{f}(t, x)$

This integrator evaluates  $\underline{f}$  twice and hence the terminology "two function integration". It provides third order integration of  $\underline{x}$  and  $\dot{x}$ . If  $\ddot{x} = \underline{f}(t)$ , the integration will still only be third order for  $\underline{x}$  and  $\dot{x}$ . The integration parameters are due to Nyström, Reference 2, and are the solution of six constraint equations in six unknowns, which are given in Reference 1.

$$k_1 = \Delta T \underline{f}(t_n, x_n)$$

$$k_2 = \Delta T \underline{f}(t_n + \delta_2 \Delta T, x_n + \delta_2 \Delta T \dot{x}_n + a_1 \Delta T k_1)$$

$$x_{n+1} = x_n + \Delta T \dot{x}_n + \Delta T (\alpha_1 k_1 + \alpha_2 k_2) + O(\Delta T^4)$$

$$\dot{x}_{n+1} = \dot{x}_n + \beta_1 k_1 + \beta_2 k_2 + O(\Delta T^4)$$

Where  $\Delta T = t_{n+1} - t_n$  and the integration coefficients are given by

$$\delta_2 = 2/3$$

$$a_1 = 2/9$$

$$\alpha_1 = 1/4$$

$$\alpha_2 = 1/4$$

$$\beta_1 = 1/4$$

$$\beta_2 = 3/4$$

### 3. THREE FUNCTION INTEGRATION OF $\ddot{x} = f(t, x)$

This integrator provides fourth order integration for both  $x$  and  $\dot{x}$ . However, if  $\ddot{x} = f(t)$ , the integrator will be fifth order for both  $x$  and  $\dot{x}$ . The integration parameters are due to Lear.

$$\begin{aligned} k_1 &= \Delta T f(t_n, x_n) \\ k_2 &= \Delta T f(t_n + \delta_2 \Delta T, x_n + \delta_2 \Delta T \dot{x}_n + a_1 \Delta T k_1) \\ k_3 &= \Delta T f[t_n + \delta_3 \Delta T, x_n + \delta_3 \Delta T \dot{x}_n + \Delta T(b_1 k_1 + b_2 k_2)] \\ \dot{x}_{n+1} &= \dot{x}_n + \Delta T \ddot{x}_n + \Delta T(\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3) \\ x_{n+1} &= x_n + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 \end{aligned}$$

The integration coefficients are given by

$$\begin{aligned} \delta_2 &= .6 - \sqrt{.06} = .35505 \ 10257 \\ \delta_3 &= .6 + \sqrt{.06} = .84494 \ 89743 \\ a_1 &= .21 - .6 \sqrt{.06} = .06303 \ 06154 \\ b_1 &= (.15 + 4 \sqrt{.06})/25 = .04519 \ 18359 \\ b_2 &= (5.1 + 11 \sqrt{.06})/25 = .31177 \ 75487 \\ \alpha_1 &= 1/9 = .11111 \ 11111 \\ \alpha_2 &= (7 + 20 \sqrt{.06})/36 = .33052 \ 72081 \\ \alpha_3 &= (7 - 20 \sqrt{.06})/36 = .05836 \ 16809 \\ \beta_1 &= 1/9 = .11111 \ 11111 \\ \beta_2 &= (8 + 5 \sqrt{.06})/18 = .51248 \ 58262 \\ \beta_3 &= (8 - 5 \sqrt{.06})/18 = .37640 \ 30627 \end{aligned}$$

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#### 4. FOUR FUNCTION INTEGRATION OF $\ddot{x} = f(t, x)$

This integrator provides fifth order integration for  $\underline{x}$  and  $\dot{\underline{x}}$ . However, if  $\ddot{x} = f(t)$ , the integrator will be seventh order for both  $\underline{x}$  and  $\dot{\underline{x}}$ . The integration parameters are due to Lear.

$$\underline{k}_1 = \Delta T f(t_n, \underline{x}_n)$$

$$\underline{k}_2 = \Delta T f(t_n + \delta_2 \Delta T, \underline{x}_n + \delta_2 \Delta T \dot{\underline{x}}_n + a_1 \Delta T \underline{k}_1)$$

$$\underline{k}_3 = \Delta T f[t_n + \delta_3 \Delta T, \underline{x}_n + \delta_3 \Delta T \dot{\underline{x}}_n + \Delta T(b_1 \underline{k}_1 + b_2 \underline{k}_2)]$$

$$\underline{k}_4 = \Delta T f[t_n + \delta_4 \Delta T, \underline{x}_n + \delta_4 \Delta T \dot{\underline{x}}_n + \Delta T(c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3)]$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta T \dot{\underline{x}}_n + \Delta T(\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2 + \alpha_3 \underline{k}_3 + \alpha_4 \underline{k}_4)$$

$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_n + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \beta_3 \underline{k}_3 + \beta_4 \underline{k}_4$$

∴ . The integration coefficients were obtained by numerically solving 17 non-linear equations in 17 unknowns. The solution is

$$\delta_2 = .21234 \ 05385$$

$$\delta_3 = .59053 \ 31358$$

$$\delta_4 = .91141 \ 20406$$

$$a_1 = .022544 \ 25214$$

$$b_1 = -.00114 \ 39805$$

$$b_2 = .17550 \ 86728$$

$$c_1 = .11715 \ 41673$$

$$c_2 = .13937 \ 54710$$

$$c_3 = .15880 \ 63156$$

$$\alpha_1 = .06250\ 00001$$

$$\alpha_2 = .25901\ 73402$$

$$\alpha_3 = .15895\ 23623$$

$$\alpha_4 = .01953\ 02974$$

$$\beta_1 = .06250\ 00001$$

$$\beta_2 = .32884\ 43202$$

$$\beta_3 = .38819\ 34687$$

$$\beta_4 = .22046\ 22110$$

Note that probably both  $\alpha_1$  and  $\beta_1$  are exactly  $.0625 = 1/16$ . Computer roundoff error probably prevented the correct solution. However, the above set of coefficients should be used as shown, since they satisfy many of the defining constraint equations exactly.

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## 5. FIVE FUNCTION INTEGRATION OF $\ddot{x} = f(t, x)$ - - -

This integrator provides sixth order integration for  $x$  and  $\dot{x}$ . There are 22 constraint equations in 24 unknowns, thus two more constraint equations were chosen so as to cause eighth order integration of  $\dot{x} = f(t)$ . Unfortunately, the numerical solutions for the new constraint equations were unsatisfactory. Thus the integration constants shown here are only good to sixth order for integrating  $\ddot{x} = f(t)$ .

$$\begin{aligned}
 k_1 &= \Delta T f(t_n, x_n) \\
 k_2 &= \Delta T f(t_n + \delta_2 \Delta T, x_n + \delta_2 \Delta T \dot{x}_n + a_1 \Delta t k_1) \\
 k_3 &= \Delta T f[t_n + \delta_3 \Delta T, x_n + \delta_3 \Delta T \dot{x}_n + \Delta T (b_1 k_1 + b_2 k_2)] \\
 k_4 &= \Delta T f[t_n + \delta_4 \Delta T, x_n + \delta_4 \Delta T \dot{x}_n + \Delta T (c_1 k_1 + c_2 k_2 + c_3 k_3)] \\
 k_5 &= \Delta T f[t_n + \delta_5 \Delta T, x_n + \delta_5 \Delta T \dot{x}_n + \Delta T (d_1 k_1 + d_2 k_2 + d_3 k_3 + d_4 k_4)] \\
 x_{n+1} &= x_n + \Delta T \dot{x}_n + \Delta T (\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 + \alpha_5 k_5) \\
 \dot{x}_{n+1} &= \dot{x}_n + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3 + \beta_4 k_4 + \beta_5 k_5
 \end{aligned}$$

The integration coefficients, due to Lear, are

$$\begin{aligned}
 \delta_2 &= \frac{1}{2} & \delta_3 &= \frac{1}{3} & \delta_4 &= \frac{2}{3} & \delta_5 &= 1 \\
 a_1 &= \frac{1}{8} & b_1 &= \frac{1}{18} & b_2 &= 0 \\
 c_1 &= \frac{1}{9} & c_2 &= 0 & c_3 &= \frac{1}{9} \\
 d_1 &= 0 & d_2 &= -\frac{8}{11} & d_3 &= \frac{9}{11} & d_4 &= \frac{9}{22} \\
 \alpha_1 &= \frac{11}{120} & \alpha_2 &= -\frac{4}{15} & \alpha_3 &= \frac{9}{20} & \alpha_4 &= \frac{9}{40} & \alpha_5 &= 0 \\
 \beta_1 &= \frac{11}{120} & \beta_2 &= -\frac{8}{15} & \beta_3 &= \beta_4 = \frac{27}{40} & \beta_5 &= \frac{11}{120}
 \end{aligned}$$

Monuki's integration coefficients (Reference 1) are

$$\delta_2 = .3 \quad \delta_3 = .6 \quad \delta_4 = 2/3 \quad \delta_5 = 1$$

$$a_1 = .045 \quad b_1 = .18 \quad b_2 = 0$$

$$c_1 = .13671 \ 69639$$

$$c_2 = .08047 \ 55373$$

$$c_3 = .00502 \ 97211$$

$$d_1 = .00740 \ 74074$$

$$d_2 = .49023 \ 56902$$

$$d_3 = -.57037 \ 03704$$

$$d_4 = .57272 \ 72727$$

$$\alpha_1 = .08796 \ 29630$$

$$\alpha_2 = .33670 \ 03367$$

$$\alpha_3 = -.23148 \ 14815$$

$$\alpha_4 = .30681 \ 81818$$

$$\alpha_5 = 0$$

$$\beta_1 = .08796 \ 29630$$

$$\beta_2 = .48100 \ 04810$$

$$\beta_3 = -.57870 \ 37037$$

$$\beta_4 = .92045 \ 45455$$

$$\beta_5 = .08928 \ 57143$$

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## 6. SIX FUNCTION INTEGRATION OF $\dot{x} = f(t, x)$

This integrator provides seventh order integration for  $x$  and  $\dot{x}$ . There were 33 constraint equations in 32 unknowns, impossible to solve at first glance. However, Monuki, Reference 1, using a brilliant set of assumptions, was able to show how to obtain a solution.

$$\begin{aligned}
 \underline{k}_1 &= \Delta T f(t_n, \underline{x}_n) \\
 \underline{k}_2 &= \Delta T f(t_n + \delta_2 \Delta T, \underline{x}_n + \delta_2 \Delta T \dot{\underline{x}}_n + a_1 \Delta T \underline{k}_1) \\
 \underline{k}_3 &= \Delta T f[t_n + \delta_3 \Delta T, \underline{x}_n + \delta_3 \Delta T \dot{\underline{x}}_n + \Delta T(b_1 \underline{k}_1 + b_2 \underline{k}_2)] \\
 \underline{k}_4 &= \Delta T f[t_n + \delta_4 \Delta T, \underline{x}_n + \delta_4 \Delta T \dot{\underline{x}}_n + \Delta T(c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3)] \\
 \underline{k}_5 &= \Delta T f[t_n + \delta_5 \Delta T, \underline{x}_n + \delta_5 \Delta T \dot{\underline{x}}_n + \Delta T(d_1 \underline{k}_1 + d_2 \underline{k}_2 + d_3 \underline{k}_3 + d_4 \underline{k}_4)] \\
 \underline{k}_6 &= \Delta T f[t_n + \delta_6 \Delta T, \underline{x}_n + \delta_6 \Delta T \dot{\underline{x}}_n + \Delta T(e_1 \underline{k}_1 + e_2 \underline{k}_2 + e_3 \underline{k}_3 + e_4 \underline{k}_4 + e_5 \underline{k}_5)] \\
 \underline{x}_{n+1} &= \underline{x}_n + \Delta T \dot{\underline{x}}_n + \Delta T(\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2 + \alpha_3 \underline{k}_3 + \alpha_4 \underline{k}_4 + \alpha_5 \underline{k}_5 + \alpha_6 \underline{k}_6) \\
 \dot{\underline{x}}_{n+1} &= \dot{\underline{x}}_n + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \beta_3 \underline{k}_3 + \beta_4 \underline{k}_4 + \beta_5 \underline{k}_5 + \beta_6 \underline{k}_6
 \end{aligned}$$

The integration coefficients, due to Monuki, are

$$\begin{aligned}
 \delta_2 &= .10654 \ 17886 \\
 \delta_3 &= .21308 \ 35772 \\
 \delta_4 &= .59267 \ 23008 \\
 \delta_5 &= .916 \\
 \delta_6 &= .972 \\
 a_1 &= .56755 \ 76359 \cdot 10^{-2} \\
 b_1 &= .07567 \ 43515 \cdot 10^{-1} \\
 b_2 &= .15134 \ 87029 \cdot 10^{-1}
 \end{aligned}$$

$c_1 = .14003\ 61674$   
 $c_2 = -.25447\ 80570$   
 $c_3 = .29007\ 21177$   
 $d_1 = -1.02164\ 36141$   
 $d_2 = 2.65397\ 01073$   
 $d_3 = -1.48615\ 90950$   
 $d_4 = .27336\ 06017$   
 $e_1 = -20.40832\ 94915$   
 $e_2 = 50.31431\ 81086$   
 $e_3 = -32.30441\ 78724$   
 $e_4 = 2.94949\ 60939$   
 $e_5 = -.07867\ 48385$   
 $\alpha_1 = .06271\ 70177$   
 $\alpha_2 = 0$   
 $\alpha_3 = .25968\ 74616$   
 $\alpha_4 = .15875\ 55586$   
 $\alpha_5 = .01912\ 37845$   
 $\alpha_6 = -.00028\ 38224$   
 $\beta_1 = .06271\ 70177$   
 $\beta_2 = 0$   
 $\beta_3 = .33000\ 64074$   
 $\beta_4 = .38974\ 89881$   
 $\beta_5 = .22766\ 41014$   
 $\beta_6 = -.01013\ 65146$

## 7. TWO FUNCTION INTEGRATION OF $\dot{\underline{x}} = \underline{f}(t, \underline{x}, \dot{\underline{x}})$

This integrator provides third order integration for  $\underline{x}$  and second order for  $\dot{\underline{x}}$ . However, if  $\dot{\underline{x}} = \underline{f}(t, \underline{x})$  or if  $\dot{\underline{x}} = \underline{f}(t)$  then the integration will be third order for both  $\underline{x}$  and  $\dot{\underline{x}}$ . The integration parameters are due to Nyström, Reference 2, and are the solution of seven constraint equations in seven unknowns.

$$\underline{k}_1 = \Delta T \underline{f}(t_n, \underline{x}_n, \dot{\underline{x}}_n)$$

$$\underline{k}_2 = \Delta T \underline{f}(t_n + \delta_2 \Delta T, \underline{x}_n + \delta_2 \Delta T \dot{\underline{x}}_n + a_1 \Delta T \underline{k}_1, \dot{\underline{x}}_n + \dot{a}_1 \underline{k}_1)$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta T \dot{\underline{x}}_n + \Delta T (\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2)$$

$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_n + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2$$

The integration coefficients are given by

$$\delta_2 = 2/3$$

$$a_1 = 2/9$$

$$\alpha_1 = 1/4$$

$$\alpha_2 = 1/4$$

$$\dot{a}_1 = 2/3$$

$$\beta_1 = 1/4$$

$$\beta_2 = 3/4$$

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### 8. THREE FUNCTION INTEGRATION OF $\ddot{x} = f(t, x, \dot{x})$

This integrator provides fourth order integration for  $x$  and third order for  $\dot{x}$ . If  $\dot{x} = f(t, x)$  the integrator will be fourth order for both  $x$  and  $\dot{x}$ . If  $\ddot{x} = f(t)$  the integrator will be fifth order for both  $x$  and  $\dot{x}$ . The integration constants are due to Lear.

$$k_1 = \Delta T f(t_n, x_n, \dot{x}_n)$$

$$k_2 = \Delta T f(t_n + \delta_2 \Delta T, x_n + \delta_2 \Delta T \dot{x}_n + a_1 \Delta T k_1, \dot{x}_n + \dot{a}_1 k_1)$$

$$k_3 = \Delta T f[t_n + \delta_3 \Delta T, x_n + \delta_3 \Delta T \dot{x}_n + \Delta T(b_1 k_1 + b_2 k_2), \dot{x}_n + \dot{b}_1 k_1 + \dot{b}_2 k_2]$$

$$x_{n+1} = x_n + \Delta T \dot{x}_n + \Delta T(\alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3)$$

$$\dot{x}_{n+1} = \dot{x}_n + \beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3$$

The integration coefficients are given by

$$\delta_2 = .6 - \sqrt{.06} = .35505 \ 10257$$

$$\delta_3 = .6 + \sqrt{.06} = .84494 \ 89743$$

$$a_1 = .21 - .6 \sqrt{.06} = .06303 \ 06154$$

$$b_1 = (.15 + 4 \sqrt{.06})/25 = .04519 \ 18359$$

$$b_2 = (5.1 + 11 \sqrt{.06})/25 = .31177 \ 75487$$

$$\alpha_1 = 1/9 = .11111 \ 11111$$

$$\alpha_2 = (7 + 20 \sqrt{.06})/36 = .33052 \ 72081$$

$$\alpha_3 = (7 - 20 \sqrt{.06})/36 = .05836 \ 16809$$

$$\dot{a}_1 = .6 - \sqrt{.06} = .35505 \ 10257$$

$$\dot{b}_1 = -(5.4 + 19 \sqrt{.06})/25 = -.40216 \ 12204$$

$$\dot{b}_2 = (20.4 + 44 \sqrt{.06})/25 = 1.24711 \ 01947$$

$$\beta_1 = 1/9 = .11111 \ 11111$$

$$\beta_2 = (8 + 5 \sqrt{.06})/18 = .51248 \ 58262$$

$$\beta_3 = (8 - 5 \sqrt{.06})/18 = .37640 \ 30627$$

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## 9. FOUR FUNCTION INTEGRATION OF $\underline{x} = \underline{f}(t, \underline{x}, \dot{\underline{x}})$

This integrator provides fourth order integration for both  $\underline{x}$  and  $\dot{\underline{x}}$ . If  $\ddot{\underline{x}} = \underline{f}(t, \underline{x})$  the integration will be fifth order for both  $\underline{x}$  and  $\dot{\underline{x}}$ . If  $\ddot{\underline{x}} = \underline{f}(t)$  the integrator will be sixth order for both  $\underline{x}$  and  $\dot{\underline{x}}$ . The integration constants are due to Lear.

$$\begin{aligned} \underline{k}_1 &= \Delta T \underline{f}(t_n, \underline{x}_n, \dot{\underline{x}}_n) \\ \underline{k}_2 &= \Delta T \underline{f}(t_n + \delta_2 \Delta T, \underline{x}_n + \delta_2 \Delta T \dot{\underline{x}}_n + a_1 \Delta T \underline{k}_1, \dot{\underline{x}}_n + \dot{a}_1 \underline{k}_1) \\ \underline{k}_3 &= \Delta T \underline{f}[t_n + \delta_3 \Delta T, \underline{x}_n + \delta_3 \Delta T \dot{\underline{x}}_n + \Delta T(b_1 \underline{k}_1 + b_2 \underline{k}_2), \dot{\underline{x}}_n + \dot{b}_1 \underline{k}_1 + \dot{b}_2 \underline{k}_2] \\ \underline{k}_4 &= \Delta T \underline{f}[t_n + \delta_4 \Delta T, \underline{x}_n + \delta_4 \Delta T \dot{\underline{x}}_n + \Delta T(c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3), \dot{\underline{x}}_n \\ &\quad + \dot{c}_1 \underline{k}_1 + \dot{c}_2 \underline{k}_2 + \dot{c}_3 \underline{k}_3] \\ \underline{x}_{n+1} &= \underline{x}_n + \Delta T \dot{\underline{x}}_n + \Delta T(\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2 + \alpha_3 \underline{k}_3 + \alpha_4 \underline{k}_4) \\ \dot{\underline{x}}_{n+1} &= \dot{\underline{x}}_n + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \beta_3 \underline{k}_3 + \beta_4 \underline{k}_4 \end{aligned}$$

The integration coefficients are given by

$$\begin{aligned} \delta_2 &= (5 - \sqrt{5})/10 = .27639 \ 32023 \\ \delta_3 &= (5 + \sqrt{5})/10 = .72360 \ 67977 \\ \delta_4 &= 1. \\ a_1 &= (3 - \sqrt{5})/20 = .038196 \ 60115 \\ b_1 &= 0 \\ b_2 &= (3 + \sqrt{5})/20 = .26180 \ 33989 \\ c_1 &= (-1 + \sqrt{5})/4 = .30901 \ 69943 \\ c_2 &= 0 \\ c_3 &= (3 - \sqrt{5})/4 = .19098 \ 30058 \end{aligned}$$

$$\begin{aligned}
\alpha_1 &= 1/12 = .08333\ 33333 \\
\alpha_2 &= (5+\sqrt{5})/24 = .30150\ 28324 \\
\alpha_3 &= (5-\sqrt{5})/24 = .11516\ 38343 \\
\alpha_4 &= 0 \\
\dot{a}_1 &= (5-\sqrt{5})/10 = .27639\ 32023 \\
\dot{b}_1 &= -(5+3\sqrt{5})/20 = -.58541\ 01966 \\
\dot{b}_2 &= (3+\sqrt{5})/4 = 1.30901\ 69944 \\
\dot{c}_1 &= -(1-5\sqrt{5})/4 = 2.54508\ 49719 \\
\dot{c}_2 &= -(5+3\sqrt{5})/4 = -2.92705\ 09831 \\
\dot{c}_3 &= (5-\sqrt{5})/2 = 1.38196\ 60113 \\
\beta_1 &= 1/12 = .08333\ 33333 \\
\beta_2 &= 5/12 = .41666\ 66667 \\
\beta_3 &= 5/12 = .41666\ 66667 \\
\beta_4 &= 1/12 = .08333\ 33333 \quad \text{_____}
\end{aligned}$$

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## 10. INTEGRATION OF $\ddot{\underline{x}} = \underline{f}(t, \underline{x}, \dot{\underline{x}}) + \underline{\epsilon}(t, \underline{x}, \dot{\underline{x}})$

In many problems there may be a very small perturbing acceleration,  $\underline{\epsilon}(t, \underline{x}, \dot{\underline{x}})$ , adding to a dominate acceleration,  $\underline{f}(t, \underline{x}, \dot{\underline{x}})$ . For example,  $\underline{\epsilon}(t)$  could be a small venting acceleration on an orbiting vehicle.  $\underline{\epsilon}(\underline{x})$  could be due to higher order harmonics in the earth's gravity field.  $\underline{\epsilon}(\dot{\underline{x}})$  could be due to small drag or solar pressure forces acting on a space vehicle.

When  $\underline{\epsilon}$  is time consuming to evaluate, it may be desirable to not evaluate  $\underline{\epsilon}$  as often as  $\underline{f}$  is evaluated in the integration step. Or, we may desire to use an integrator for  $\underline{x} = \underline{f}(t, \underline{x})$  when  $\underline{\epsilon}$  has velocity terms in it. In these cases we may try the following scheme.

Integrate  $\ddot{\underline{x}} = \underline{f}(t, \underline{x}, \dot{\underline{x}})$  or  $\ddot{\underline{x}} = \underline{f}(t, \underline{x})$  normally. Then to the solution add

$$\underline{x}_{n+1} = \underline{x}_{n+1} + \frac{\Delta T^2}{2} \underline{\epsilon}(t_{n+.5}, \underline{x}_{n+.5}, \dot{\underline{x}}_{n+.5})$$

$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_{n+1} + \frac{\Delta T}{1} \underline{\epsilon}(t_{n+.5}, \underline{x}_{n+.5}, \dot{\underline{x}}_{n+.5})$$

where, we see,  $\underline{\epsilon}$  is evaluated at the midpoint of the integration step. The equations for the midpoint are

$$t_{n+.5} = t_n + \Delta T/2$$

$$\underline{x}_{n+.5} = \underline{x}_n + \frac{1}{2} (\underline{x}_{n+1} - \underline{x}_n) - \frac{\Delta T}{8} (\dot{\underline{x}}_{n+1} - \dot{\underline{x}}_n) + O(\Delta T^4)$$

$$\dot{\underline{x}}_{n+.5} = \frac{3}{2\Delta T} (\underline{x}_{n+1} - \underline{x}_n) - \frac{1}{4} (\dot{\underline{x}}_{n+1} - \dot{\underline{x}}_n) - \frac{1}{2} \dot{\underline{x}}_n + O(\Delta T^4)$$

$$= (\dot{\underline{x}}_{n+1} + \dot{\underline{x}}_n)/2 \text{ for } \Delta T = 0 \text{ or } \Delta T \text{ small}$$

Note that the equation for  $\dot{x}_{n+1.5}$  is afflicted with roundoff error for small  $\Delta T$ . A more stable form is

$$\dot{x}_{n+1.5} = \dot{x}_n + \frac{\Delta T}{2} [\dot{x}_n + (\alpha_1 k_1 + \alpha_2 k_2 + \dots) - .25(\beta_1 k_1 + \beta_2 k_2 + \dots)]$$

$$\dot{x}_{n+1.5} = \dot{x}_n + 1.5(\alpha_1 k_1 + \alpha_2 k_2 + \dots) - .25(\beta_1 k_1 + \beta_2 k_2 + \dots)$$

where the  $\alpha_1$ ,  $\beta_1$  and  $k_1$  are those used in the Nyström integrator.

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TRW DSSG  
One Space Park  
Redondo Beach, Ca. 90278
2. E.J. Nyström, "Über die Numerische Integration von Differential Gleichungen", Acta Soc. Sci. Fenn. 50 (1925), No. 13, 1-55.

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