Equations of 3D Projections

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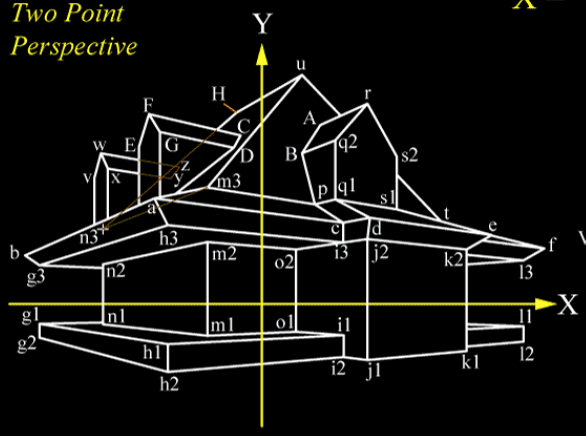
Two Point Perspective



$$X = \frac{F\,x}{y} \qquad Y = \frac{F\,z}{y}$$

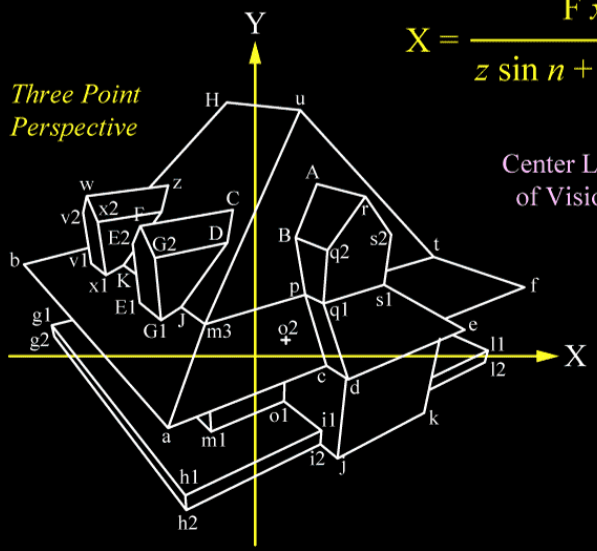
F = 10 $n = 0°$

Side View (Elevation)

Heights

N = -0.26
O = -0.44
P = 0.51
Q = 0.64
R = 0.96
S = 1.50
T = 1.73
U = 1.93
V = 2.58

Three Point Perspective

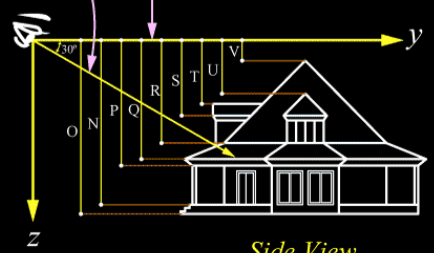


$$X = \frac{F\,x}{z\sin n + y\cos n} \qquad Y = \frac{F\,(z\cos n - y\sin n)}{z\sin n + y\cos n}$$

Center Line of Vision

Eye Level

F = 10 $n = -30°$



Side View (Elevation)

Heights

N = -3.25
O = -3.43
P = -2.48
Q = -2.35
R = -2.03
S = -1.49
T = -1.26
U = -1.06
V = -0.40

Equations of 3D Projections

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Abstract

Two simple vector equations to plot any 3D model are based on actual camera snapshot rather than an approximation of points in a Matrix.

The process involved relies on a point tracer, which means it follows any point tracing any object placed within the universal 3D coordinate system, projected by one light ray to the camera lens. What's new is incorporating the tilt angle of the camera (denoted by n°) that evidently causes the so-called bird's eye views or worm's eye views i.e. three point perspectives, that has never been considered in all procedures related to perspective projections.

The equations also include another element, that is, the zooming in/out constant (denoted by F for focal length). This allows us to magnify or reduce the size of the resulting perspective projection regardless of the inputted object's coordinates.

These scenes are called projections because they are flat two dimensional images that look like 3D scenes with depth. Depending on the location of the origin point (0), the equations can be simple or complicated. Through experimenting of multiple locations, I concluded that the simplest equations are obtained by setting the camera lens as the origin point for the object's coordinates (x,y,z); and setting the focal point on the projection plane as the origin point of the resulting perspective projection coordinates (X,Y).

The diagram used herein to prove the equations is based on the "mechanical method" developed for my own apparatus for drawing perspective views, which was patented in England in 1981 (UK Patent Application #2069938A), and the equations were listed in the patent application.

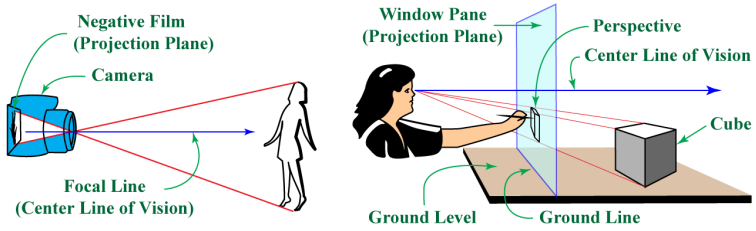
The perspective projections appear different in relation to the tilt angle of the camera (elevation or depression angle n°), for which one-, two- or three-point perspective views can be seen in photos. If you look at the photos in Fig. 01 you will easily notice that the camera was tilted downwards (depression angle for bird's eye view) or tilted upwards (elevation angle for worm's eye view) at certain angles. Thus, the tilt angle of the camera (n°) must be involved in the equations.

Fig. 01. Bird's- and Worm's-eye views

1– Projection Plane (Picture Plane)

Projection plane is in some likeness of a negative film inside a camera on which an image appears. Projection plane can also be described as a window pane from which a person standing at arm's length traces upon the glass the scene outside. A person doing this will be drawing a picture in perspective (i.e. a 3D projection), as shown in Fig. 02.

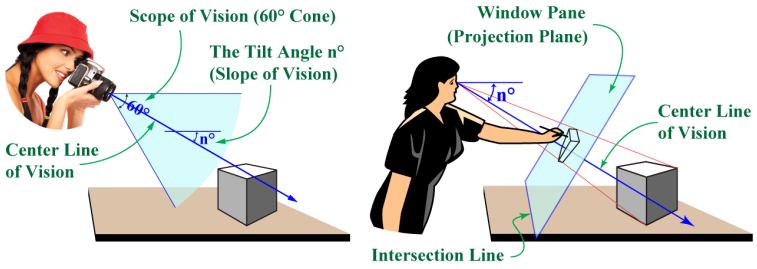
If the camera is tilted, then, its negative film (i.e. Projection Plane) will be tilted too. Thus, the window pane will be tilted as well, as it represents the negative film as a projection plane. Fig. 03.



a. *A perspective drawing i.e. a photograph is projected inside the camera.*

b. *A perspective is projected on the window pane by means of light rays coming to the eye.*

Fig. 02. Basic Interpretation of the Perspective Projection and Projection Plane.



a. *Tilted negative film (Projection Plane)*

b. *Tilted window pane (Projection Plane)*

Fig. 03. If the camera is tilted, Projection Plane must be tilted.

2– Setting up the coordinates

There are two sets of vector coordinate systems. One is a 3D coordinate system of (x,y,z) coordinates, where the object is placed in, with respect to the camera location or the observer's eye. The second system is a 2D system of (X,Y) coordinates for the resulting perspective projection. The simplest result obtained is by considering the camera lens – or the observer's eye – as the origin point (0) of (x,y,z) coordinates of the object, and the focal point at the projection plane to define the (X,Y) coordinates of the perspective projection. This was done by experimenting with many different locations of the origin point, in order to establish the simplest equations. Therefore, I will only demonstrate the simplest setting, in which the object's coordinate-origin is at the camera lens (or the observer's eye); whereas, the perspective projection origin point is at the focal point which is at the center line of vision. See Fig. 04.

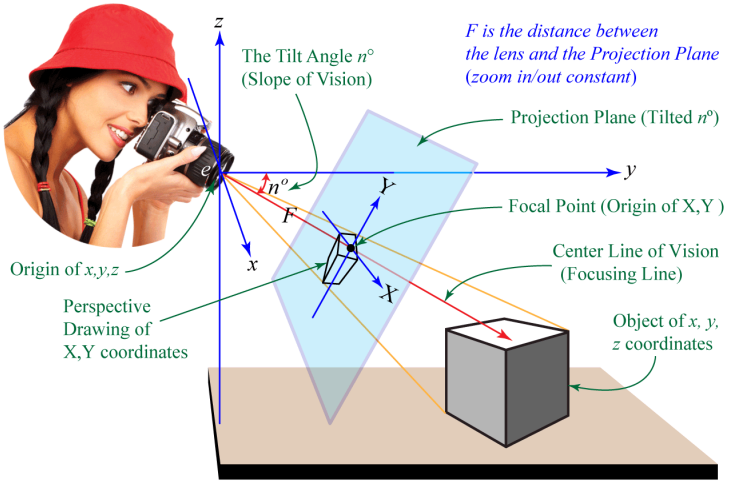


Fig. 04. Setting up the object's coordinates and the perspective coordinates.

I also considered Bird's Eye View (i.e. angle of depression) as a negative (n°); whereas, Worm's Eye View (i.e. angle of elevation) as a positive (n°), as measured from an imaginary horizontal line. For instance, (n = -30°) yields a bird's eye view; whereas, (n = 30°) yields a worm's eye view for the same coordinates of the object. When the camera is set level, (n°) becomes (0°). In perspective drawing, one– and two–point perspective projections can be obtained by setting (n = 0°). Whereas, three–point perspective projections (i.e. bird's– and worm's– eye views) are obtained by setting (-90° < n° < 90°). That's the range of n°. (F) is the focal length, used to define the required size of the perspective projection.

3– Movement of a Point in space

Fig. 05, is a 3D illustration, where point (a) is placed on (x,y) plane at (z) distance from the eye level (i.e. y-axis) and at (y) distance from the eye (e) (i.e. z-axis). Now, the light ray that comes from (a) to the eye intersects the Projection Plane defining (a1) which is the perspective representation of point (a). Thus, the gap between (a1) and X-axis is the Y-coordinate of point (a1). You may wonder why point (h) is used to define the relation between x in the plan and X in perspective on the Projection Plane. Well, if (a) as part of the triangle (aeh) is rotated about (eh) which is parallel to the Projection Plane, the intersection lines on the Projection Plane will be vertical lines. Therefore, (a1) will have the same X-coordinate as (f) in perspective. (x,y) Plane will be shifted up and down as point (a) vertically moves. Thus, (h) will also move. Therefore, the distance (ah) is variable as the height (z) changes. Consequently, X-coordinate varies as (z) changes. On the other hand, if (a) moves horizontally on its (x,y) plane in any direction, then, its (x,y) coordinates will change accordingly. However, (h) will not be affected by (x,y) movement. However, (x,y) plane will be shifted up or down as z-coordinate changes. The dynamic motion of any point, such as (a) shifts the (x,y) plane up or down as it changes its height from the eye level causing (h) to move, and this gives the effect of perspective.

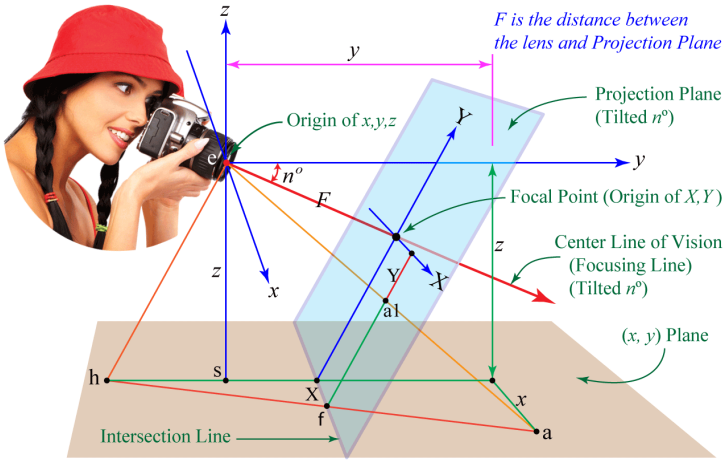


Fig. 05. The perspective procedure used to locate point (a1) on the Projection Plane.

The proof of these equations is as follows:

In Plane Geometry (refer to Fig. 06):

$$\frac{bk}{z} = \tan n \quad \text{Thus} \quad bk = z \tan n$$

$$\frac{z}{ak} = \cos n \quad \text{Thus} \quad ak = \frac{z}{\cos n}$$

$$\frac{F}{ec} = \cos n \quad \text{Thus} \quad ec = \frac{F}{\cos n}$$

$$\frac{gc}{F} = \tan n \quad \text{Thus} \quad gc = F \tan n$$

$$\frac{dc}{ak} = \frac{ec}{ek} \quad \text{Thus} \quad dc = \frac{(ak)(ec)}{ek}$$

$$Y = gc - dc \quad \text{Thus} \quad Y = gc - \frac{(ak)(ec)}{ek}$$

Substituting in,

$$Y = F \tan n - \frac{\left(\frac{z}{\cos n}\right)\left(\frac{F}{\cos n}\right)}{y + z \tan n}$$

$$Y = \frac{F \tan n(y + z \tan n) - \frac{Fz}{\cos^2 n}}{y + z \tan n}$$

$$Y = \frac{Fy \sin n \cos n + Fz \sin^2 n - Fz}{y \cos^2 n + z \sin n \cos n}$$

Observe that (z) and (n) are negative, thus,

$$\sin n = -\sin(-n) \quad \text{and} \quad \cos n = \cos(-n)$$

$$Y = \frac{-Fy \sin n \cos n - Fz \sin^2 n + Fz}{y \cos^2 n + z \sin n \cos n}$$

$$Y = \frac{Fz(1 - \sin^2 n) - Fy \sin n \cos n}{\cos n\,(y \cos n + z \sin n)}$$

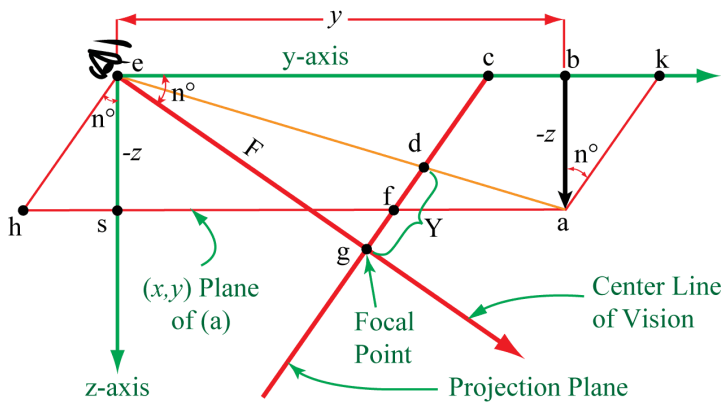
$$Y = \frac{Fz \cos^2 n - Fy \sin n \cos n}{\cos n\,(y \cos n + z \sin n)}$$

$$Y = \frac{F(z \cos n - y \sin n)}{z \sin n + y \cos n}$$

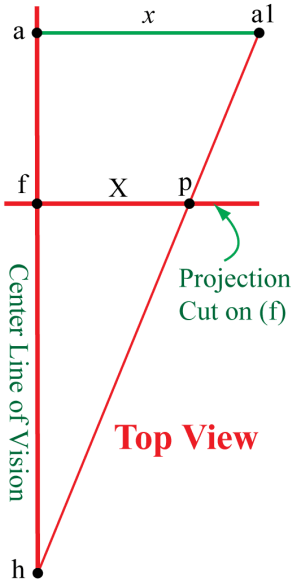
In the Top View,

$$\frac{X}{x} = \frac{fh}{ah} \quad \text{Thus,} \quad X = \frac{x(fh)}{ah}$$

$$X = \frac{\frac{xF}{\cos n}}{y + z \tan n} \qquad X = \frac{Fx}{z \sin n + y \cos n}$$



Side View



Top View

Where:

(x) is the x-coordinate of a point (a) in the plan from the Center Line of Vision.

(y) is the y-coordinate of a point (a) in the plan from the Station Point (s).

(z) is the z-coordinate of a point (a) in the elevation from the Eye Level.

(X) is the X-coordinate of a corresponding point in perspective from the Center Line of Vision (or Focal Point).

(Y) is the Y-coordinate of a corresponding point in perspective from the Focal Level (g).

(F) is the Focal Length (a magnification factor).

(n°) is the tilt angle of the camera.

Fig. 06. Top and Side Views showing the representation of (a) on the Projection Plane as (d).

4– How to apply the equations?

I was frequently asked how to apply these equations. All you need, most importantly, is the top view (the plan) of the object, to be set on top of two crossed lines, as shown in Fig. 07. The side view is used to define the camera height and direction (i.e. tilt angle nº) in a form of a line towards the object. The z-coordinates (heights) are also measured in the elevation. To avoid distortion resulting from the camera being too close to the object, the object's height must be within the scope of vision (60º cone). The camera lens (the eye) is the origin (0) of the object's coordinates (*x*, *y*, *z*).

In the top view of Fig. 07, the station point (s) defines how far the camera is from the object. (F) is zoom in/out value. It gives you any perspective size required. It is defined by the distance between the camera and the projection plane. The bigger (F) is, the bigger the perspective projection is.

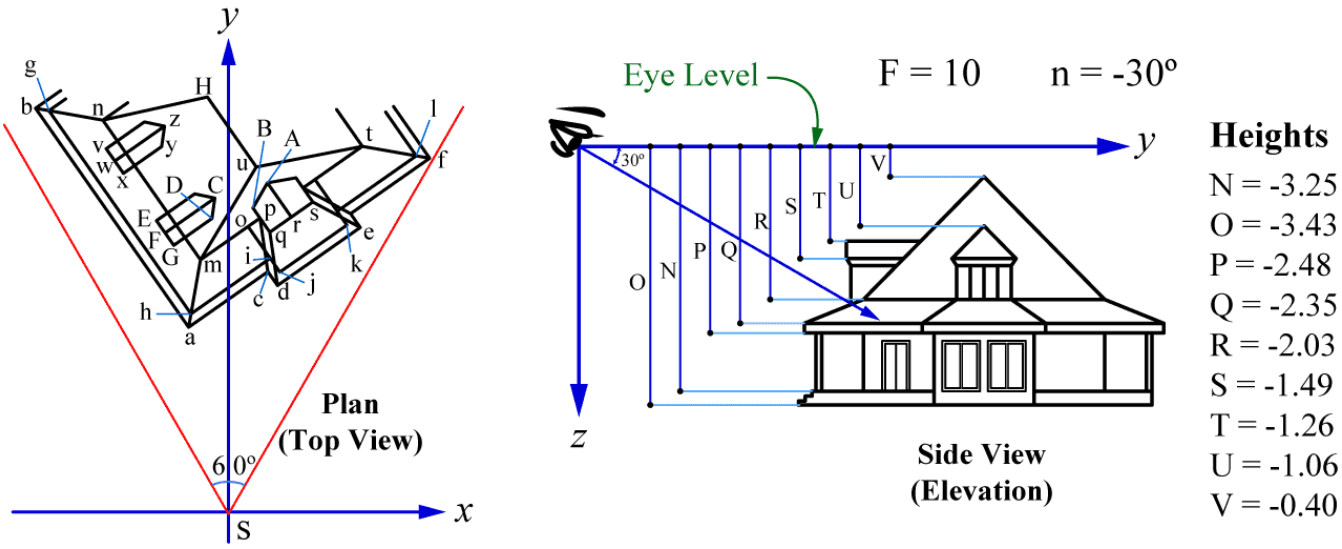


Fig. 07. Plan and elevation are used to measure (x,y,z) coordinates.

I must emphasize that negative and positive values are crucial. According to basic geometry, for y-coordinates, any measurement above the x-axis is taken positive, and below the x-axis is negative. for x-coordinates, any measurement to the right of the y-axis is taken positive, and to the left of the y-axis is negative. For z-coordinates, any measurement above the y-axis is positive, whereas below the y-axis, is negative. Similarly, for the tilt angle of the camera (nº), tilting downwards (for bird's - eye view) is a negative angle (-nº), whereas, for worm's - eye view (tilting upwards), the camera tilt angle must be set "positive" (+nº). If (nº) is set to (0), the resulting perspective view will be a commonly known as one- or two- point perspective.

When nº = 0º, the equations become (X = Fx / y) and (Y = Fz / y), since (sin 0 = 0) and (cos 0 = 1).

The range of the tilt angle (nº) is (–90º ≤ nº ≤ +90º).

With that being considered, you can use my Microsoft Excel file to input (x,y,z) values and defining at once (nº and F). The output (X,Y) coordinates of the perspective drawing will be automatically given. Then, just draw two crossed lines on a blank sheet of paper, and plot the (X,Y) points of the perspective view.

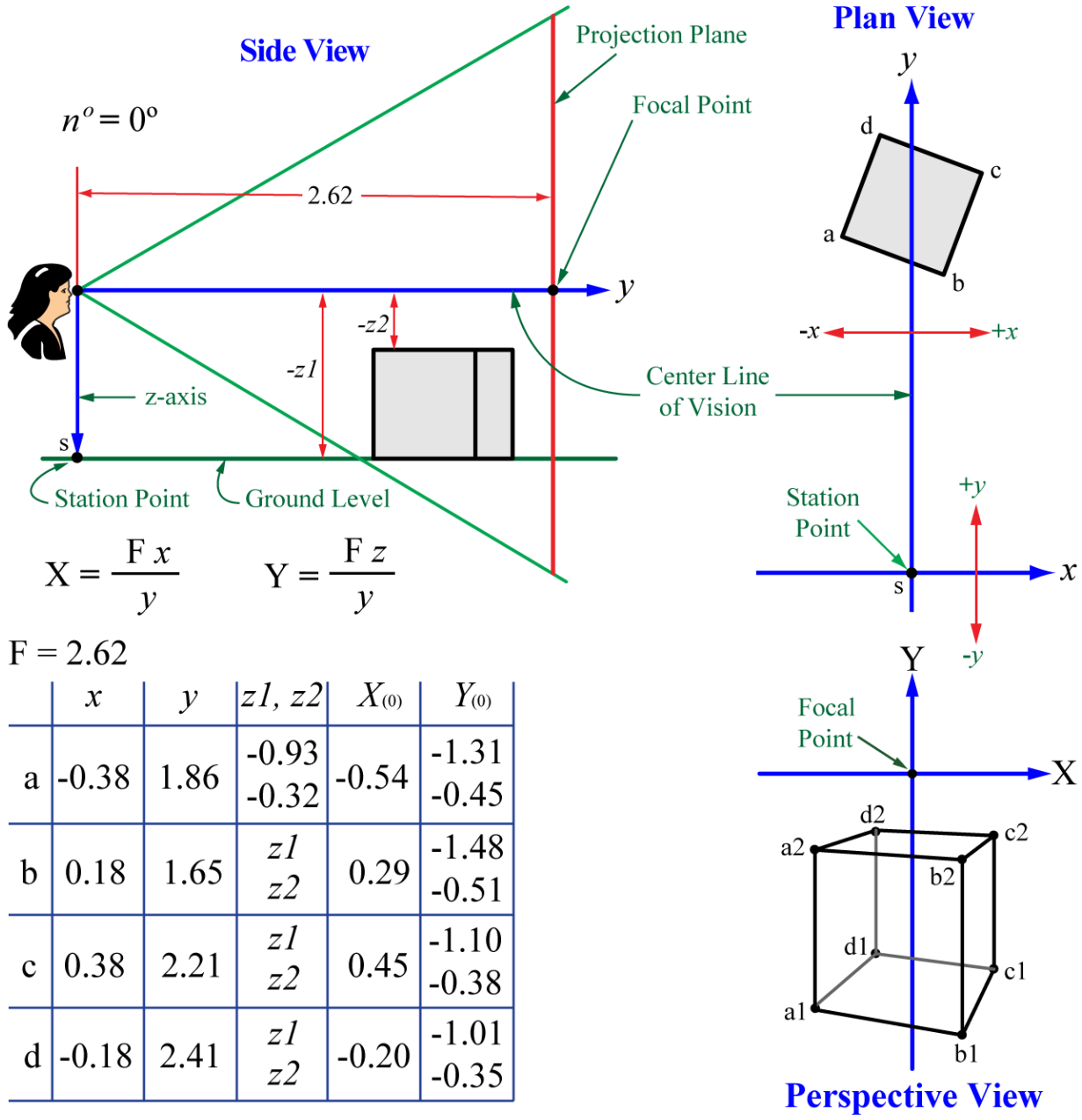
You can download my Excel file here:

https://drive.google.com/file/d/14fczx3h4TosEAYVCuJ9reCpEIqPGb_9y/view?usp=drivesdk

5– Applying the Equations to Draw a Cube (n = 0°)

In Fig. 08, below, the plan of the cube (abcd) is positioned with respect to the station point (s) which is the origin (0) of (x,y) coordinates, and the (x and y) coordinates are recorded in the table, below. There are two altitudes, (z1 = -0.93) and (z2 = -0.32) which are involved in the equation (Y = Fz/y); whereas the other equation does not have z-coordinate. You can assume any value for (F). (F) defines the position of the Projection Plane. The greater (F) is, the bigger the resulting perspective is. Plotting (X and Y) coordinates of eight points taken from the table, the perspective drawing of the cube is defined by connecting all the vertices.

When nº = 0º, (X = Fx / y) and (Y = Fz / y), since (sin 0 = 0) and (cos 0 = 1).



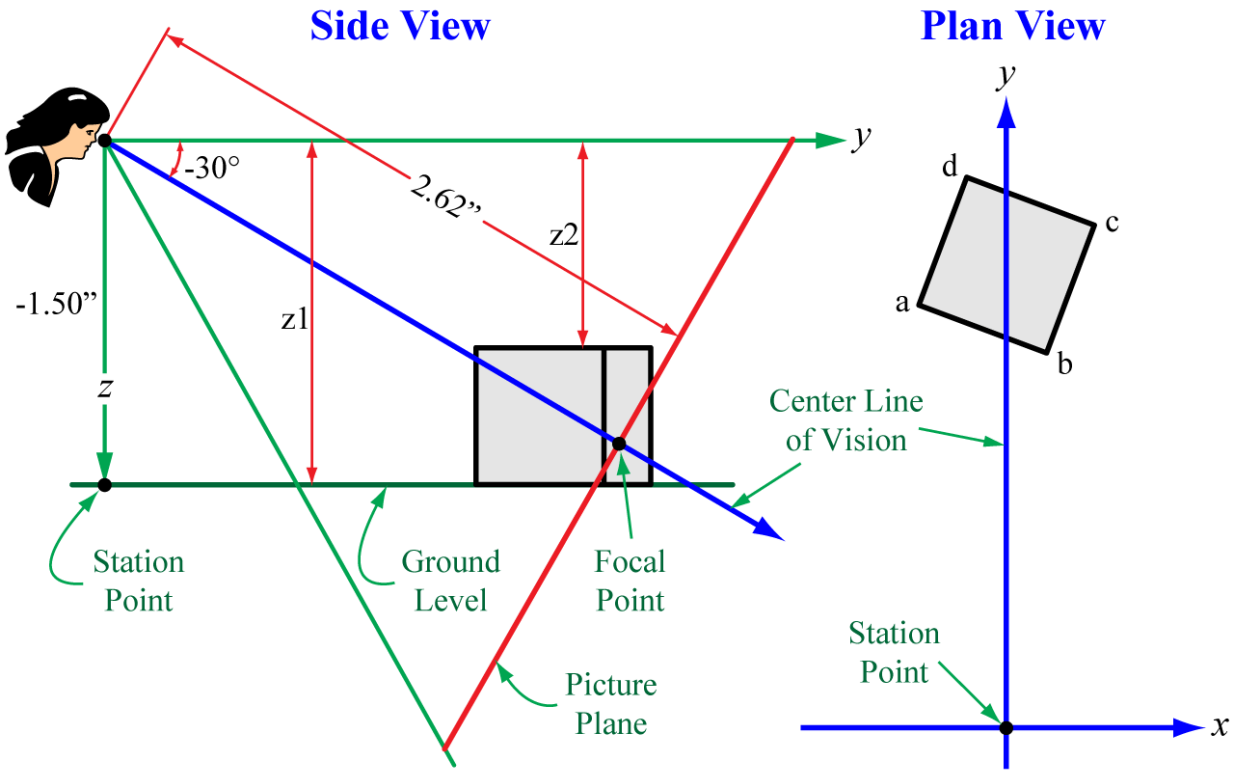
F = 2.62

	x	y	z1, z2	$X_{(0)}$	$Y_{(0)}$
a	-0.38	1.86	-0.93 -0.32	-0.54	-1.31 -0.45
b	0.18	1.65	z1 z2	0.29	-1.48 -0.51
c	0.38	2.21	z1 z2	0.45	-1.10 -0.38
d	-0.18	2.41	z1 z2	-0.20	-1.01 -0.35

Fig. 08. A cube is drawn using the equations to locate the vertices.

6– Applying the Equations to Draw a Cube (n = –30°)

In Fig. 09, below, the cube's coordinates (x, y, z) were entered in Microsoft Excel file which is programmed to execute the equations and output the perspective coordinates (X and Y) instantly. You can find a link to download this Excel file on page 5.



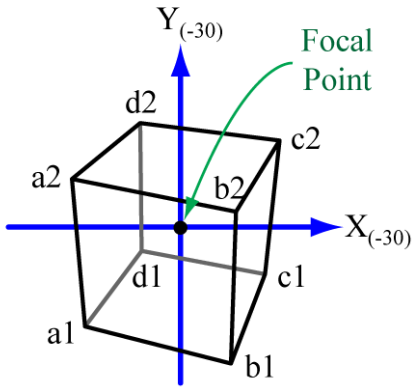
F = 2.62" and $n = -30°$

	x	y	z1, z2	$X_{(-30)}$	$Y_{(-30)}$
a	-0.38	1.86	-1.50 -0.89	-0.42 -0.48	-0.42 0.20
b	0.18	1.65	-1.50 -0.89	0.22 0.25	-0.58 0.08
c	0.38	2.21	-1.50 -0.89	0.37 0.42	-0.19 0.37
d	-0.18	2.41	-1.50 -0.89	-0.17 -0.19	-0.09 0.45

Table of Coordinates

$$X = \frac{F\,x}{z \sin n + y \cos n}$$

$$Y = \frac{F\,(z \cos n - y \sin n)}{z \sin n + y \cos n}$$



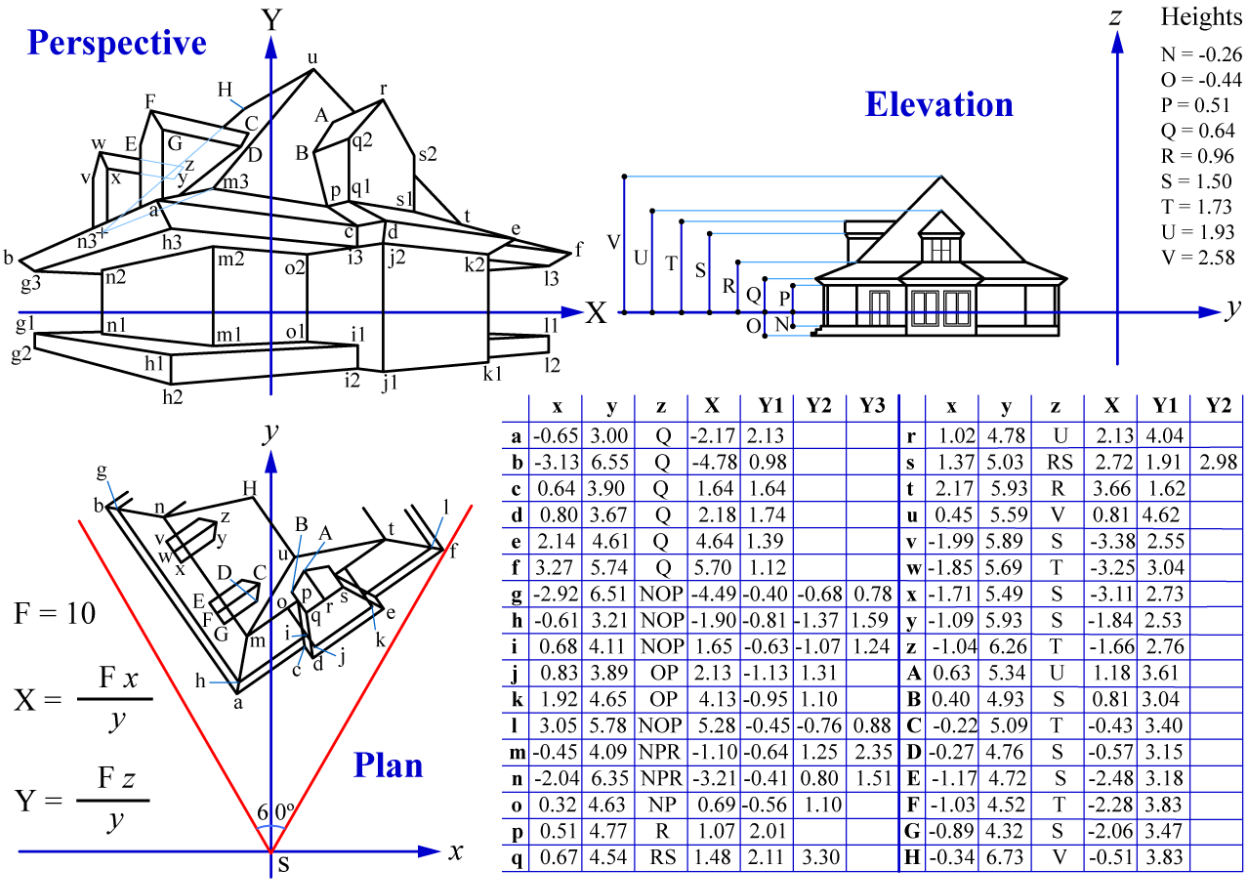
Perspective View

Fig. 09. Using the equations to draw a three-point perspective of a cube, where the tilt angle is (-30º).

7– Applying the Equations to Draw a House (n = 0°)

In Fig. 10, below, observe that any point in space can be defined by these equations, whether it is a part of a straight line or a curved line. The perspective view is plotted by the points of (X and Y) coordinates calculated by the equations, where (x and y) coordinates are taken from the plan, whereas z-coordinates are taken from the elevation.

If you are a professional drafts-person who does artistic renderings of architectural projects you will definitely appreciate these equations, as they will be a truly time saving procedure. You do not have to plot every point in perspective, but you can, at least, define the outline or any necessary spot in perspective precisely.



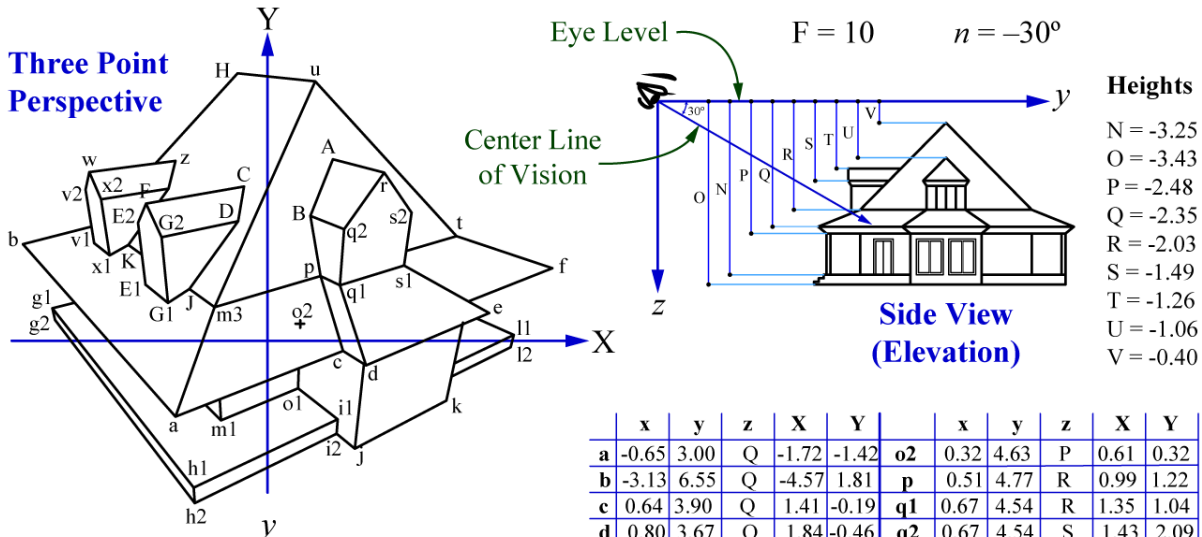
	x	y	z	X	Y1	Y2	Y3		x	y	z	X	Y1	Y2
a	-0.65	3.00	Q	-2.17	2.13			**r**	1.02	4.78	U	2.13	4.04	
b	-3.13	6.55	Q	-4.78	0.98			**s**	1.37	5.03	RS	2.72	1.91	2.98
c	0.64	3.90	Q	1.64	1.64			**t**	2.17	5.93	R	3.66	1.62	
d	0.80	3.67	Q	2.18	1.74			**u**	0.45	5.59	V	0.81	4.62	
e	2.14	4.61	Q	4.64	1.39			**v**	-1.99	5.89	S	-3.38	2.55	
f	3.27	5.74	Q	5.70	1.12			**w**	-1.85	5.69	T	-3.25	3.04	
g	-2.92	6.51	NOP	-4.49	-0.40	-0.68	0.78	**x**	-1.71	5.49	S	-3.11	2.73	
h	-0.61	3.21	NOP	-1.90	-0.81	-1.37	1.59	**y**	-1.09	5.93	S	-1.84	2.53	
i	0.68	4.11	NOP	1.65	-0.63	-1.07	1.24	**z**	-1.04	6.26	T	-1.66	2.76	
j	0.83	3.89	OP	2.13	-1.13	1.31		**A**	0.63	5.34	U	1.18	3.61	
k	1.92	4.65	OP	4.13	-0.95	1.10		**B**	0.40	4.93	S	0.81	3.04	
l	3.05	5.78	NOP	5.28	-0.45	-0.76	0.88	**C**	-0.22	5.09	T	-0.43	3.40	
m	-0.45	4.09	NPR	-1.10	-0.64	1.25	2.35	**D**	-0.27	4.76	S	-0.57	3.15	
n	-2.04	6.35	NPR	-3.21	-0.41	0.80	1.51	**E**	-1.17	4.72	S	-2.48	3.18	
o	0.32	4.63	NP	0.69	-0.56	1.10		**F**	-1.03	4.52	T	-2.28	3.83	
p	0.51	4.77	R	1.07	2.01			**G**	-0.89	4.32	S	-2.06	3.47	
q	0.67	4.54	RS	1.48	2.11	3.30		**H**	-0.34	6.73	V	-0.51	3.83	

x is the x-coordinate of a point (a) in the plan from the centre line of vision.
y is the y-coordinate of a point (a) in the plan from the Station Point (s).
z is the z-coordinate of a point (a) in the elevation from the Eye Level i.e. Horizon Line.
X is the X-coordinate of a corresponding point in perspective from the centre line (Y-axis).
Y is the Y-coordinate of a corresponding point in perspective from the Focal Level (X-axis).
F is the Focal Length (a magnification factor - zoom in/out).

Fig. 10. A house is drawn using the equations to locate the points.

8– Applying the Equations to Draw a House (n = –30°)

This is the same house shown in Fig. 10. The same equations are used but with (n = –30º) and (F = 10).



	x	y	z	X	Y		x	y	z	X	Y
a	-0.65	3.00	Q	-1.72	-1.42	**o2**	0.32	4.63	P	0.61	0.32
b	-3.13	6.55	Q	-4.57	1.81	**p**	0.51	4.77	R	0.99	1.22
c	0.64	3.90	Q	1.41	-0.19	**q1**	0.67	4.54	R	1.35	1.04
d	0.80	3.67	Q	1.84	-0.46	**q2**	0.67	4.54	S	1.43	2.09
e	2.14	4.61	Q	4.14	0.52	**r**	1.02	4.78	U	2.18	3.15
f	3.27	5.74	Q	5.32	1.36	**s1**	1.37	5.03	R	2.55	1.41
g1	-2.92	6.51	N	-4.02	0.61	**s2**	1.37	5.03	S	2.69	2.40
g2	-2.92	6.51	O	-3.97	0.39	**t**	2.17	5.93	R	3.53	1.96
g3	-2.92	6.51	P	-4.25	1.61	**u**	0.45	5.59	V	0.89	4.86
h1	-0.61	3.21	N	-1.38	-2.75	**v1**	-1.99	5.89	S	-3.24	1.83
h2	-0.61	3.21	O	-1.36	-3.04	**v2**	-1.99	5.89	S	-3.40	2.83
h3	-0.61	3.21	P	-1.52	-1.35	**w**	-1.85	5.69	T	-3.33	3.16
i1	0.68	4.11	N	1.31	-1.47	**x1**	-1.71	5.49	S	-2.95	1.60
i2	0.68	4.11	O	1.29	-1.74	**x2**	-1.71	5.49	S	-3.11	2.65
i3	0.68	4.11	P	1.42	-0.19	**y**	-1.09	5.93	S	-1.85	2.85
j1	0.83	3.89	O	1.63	-2.02	**z**	-1.04	6.26	T	-1.72	3.37
j2	0.83	3.89	P	1.80	-0.44	**A**	0.63	5.34	U	1.22	3.40
k1	1.92	4.65	O	3.34	-1.12	**B**	0.40	4.93	S	0.80	2.34
k2	1.92	4.65	P	3.65	0.34	**C**	-0.22	5.09	T	-0.44	2.89
l1	3.05	5.78	N	4.60	0.11	**D**	-0.27	4.76	S	-0.55	2.24
l2	3.05	5.78	O	4.54	-0.12	**E1**	-1.17	4.72	S	-2.28	1.05
l3	3.05	5.78	P	4.88	1.19	**E2**	-1.17	4.72	S	-2.42	2.21
m1	-0.45	4.09	N	-0.87	-1.49	**F**	-1.03	4.52	T	-2.27	2.57
m2	-0.45	4.09	P	-0.94	-0.21	**G1**	-0.89	4.32	S	-1.86	0.71
m3	-0.45	4.09	R	-0.99	0.63	**G2**	-0.89	4.32	S	-1.98	1.94
n1	-2.04	6.35	N	-2.86	0.51	**H**	-0.34	6.73	V	-0.56	5.01
n2	-2.04	6.35	P	-3.03	1.52	**I**	0.74	9.07	N	0.78	1.81
n3	-2.04	6.35	R	-3.13	2.18	**J**	-0.71	4.45	R	-1.46	0.96
o1	0.32	4.63	N	0.57	-0.89	**K**	-1.53	5.62	R	-2.60	1.79

$$X = \frac{F\,x}{z \sin n + y \cos n}$$

$$Y = \frac{F\,(z \cos n - y \sin n)}{z \sin n + y \cos n}$$

(x) is the x-coordinate of a point (a) in the plan from the centre line of vision.
(y) is the y-coordinate of a point (a) in the plan from the Station Point (s).
(z) is the z-coordinate of a point (a) in the elevation from the Horizon Line (Eye Level).
(X) is the X-coordinate of a corresponding point in perspective from the centre line (Y-axis).
(Y) is the Y-coordinate of a corresponding point in perspective from the Focal Level (X-axis).
(F) is the Focal Length (a magnification factor - zoom in/out).
(nº) is the tilt angle of the camera.

Fig. 11. Using the equations to draw a bird's eye- view of a house.

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   Published on 3rd of Sep. 1981.

   Inventor: Dhirgham Murran

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